3 Geometric Satake Equivalence I

Motivation and Preliminary

(1) What is $\check{G}$? "Langlands dual group"

W. Thurston "On progress and proof of mathematics"

"One person's clear mental image is another's intimidation"

Goal of class is to parse this sentence:

" $\check{G}$ is the group of automorphisms of cohomology of $G(\mathbb{C}[t])$-equivariant perverse sheaves on $\underbrace{G(\mathbb{C}(t))/G(\mathbb{C}[t])}_{\text{affine Grassmannian}}$ "

→ Tannakian formalism

→ Idea of Hecke algebras

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1. Tannakian formalism

Previously, $G^{\text{loc. cpt}}_{\text{abelian}} \rightsquigarrow \widehat{G} \rightsquigarrow \widehat{\widehat{G}} = G$

Now, G affine alg. gp. $\rightsquigarrow$?

For $G^{\text{cpt.}}_{\text{non-abelian}} \overset{\text{Tannaka-Krein duality}}{\rightsquigarrow}$ cat. of unitary characters

consider $\operatorname{Rep} G$ cat of f.d rep'ns of G over K

. $\operatorname{Rep} G$ is K-linear abelian category (hom-space is k-linear vector space)

. $\operatorname{Rep} G$ is monoidal: $\otimes : \operatorname{Rep} G \times \operatorname{Rep} G \to \operatorname{Rep} G$, $V, W \to V \otimes W$

w/ associativity, unity. $k = \mathbb{1} \in \operatorname{Rep} G$

- Rep G is symmetric monoidal,

$\exists\ \tau_{V,W} : V \otimes W \cong W \otimes V$ s.t. $\tau_{WV} \circ \tau_{VW} = id_{V\otimes W}$
(canonical)

- Rep G is rigid

$\forall V \in \text{Rep } G, \exists V^* \in \text{Rep } G$

$\exists$ forgetful functor

$\omega : \text{Rep } G \to (\text{Vect}_k, \otimes)$ ← symmetric monoidal rigid

exact, fully faithful ↖ fiber functor

Defn A neutral Tannakian category is a rigid symmetric monoidal k-linear abelian category A equipped with a fiber functor

$G \rightsquigarrow \text{Rep } G$ neutral Tannakian cat.

Thm (Tannakian Formalism)

A neutral tannakian category $(A, \omega : A \to \text{Vect}_k)$ is equivalent to Rep G

Where $G = \text{Aut}^{\otimes}(\omega)$

Ex

① $\text{Vect}_k \xrightarrow{\omega = id} \text{Vect}_k \rightsquigarrow \text{Aut}(id) = \{1\}$

$V \to V$ $\quad \text{Vect}_k \cong \text{Rep}\{1\}$

② $\text{Rep } G \xrightarrow{\omega = \text{Forget}} \text{Vect}_k$

$V \longmapsto V$ $\quad \text{Aut}(\omega) = G$

$\text{Rep } G \cong \text{Rep } G$

③ $\text{Vect}_k^{\mathbb{Z}}$ (f.d.) $\longmapsto \text{Vect}_k$ (f.d.) $\quad \text{Aut}(\omega) = k^\times$

$\{V_n\}_{n\in\mathbb{Z}} \longmapsto V = \oplus V_n$

gr 1-gr $\otimes$ 1-gr = 2-gr: $V_1 \otimes V_1 \cong V_2$

$z \in k^*$ acts as z^n on V_n

$\rightsquigarrow \text{Vect}_k^{\mathbb{Z}} \cong \text{Rep}(k^\times)$

(4) $\pi_1(X,x) \circlearrowright LS(X)$ cat of local systems on $X \ni x$

$\downarrow$ Fiber at x

$$\mathrm{Vect} \quad \rightsquigarrow \quad LS \overset{?}{\simeq} \mathrm{Rep}\, \pi_1(X,x)$$
$$LS \simeq \mathrm{Rep}\, \pi_1^{alg}(X,x)$$

alg hull of π_1

Rmk main motivation for motives

$\rightsquigarrow$ descendants $\left(\begin{array}{l} \text{Galois rep'ns } (\ell\text{-adic}) \\ \text{mixed Hodge structure} \end{array} \right)$

Rmk (extensions)

- $A \to QC(S)$
- A symm monoidal $\overset{E_\infty}{\rightsquigarrow}$ $A = \mathrm{Rep}\, G$
- A braided (E_2) monoidal $\rightsquigarrow$ $A \cong \mathrm{Rep}\, U_q G$
- A E_n-monoidal $\rightsquigarrow$ $A \simeq ?$
- In DAG,

  $A \to \mathrm{Vect}_k$

  replaced by $X \mapsto QCoh(X)$

  If $X = BG = pt/G$

  $QCoh(BG) \cong \mathrm{Rep}\, G$

2. Hecke algebra

$G \rightsquigarrow (\mathcal{A}_{G,\omega})$ neutral Tannakian

$\rightsquigarrow \mathcal{A}_G \simeq \mathrm{Rep}(\mathrm{Aut}^{\otimes}(\omega))$

where $\mathrm{Aut}(\omega) = \check{G}$

<u>Goal</u>: Construct $\mathcal{A}_G$

Let's work with a finite gp. H

$(\mathbb{C}[H], *)$ group algebra

$$(\varphi_1 * \varphi_2)(h) = \sum_{x \in H} \varphi_1(x)\, \varphi_2(x^{-1}h) = \sum_{xy=h} \varphi_1(x)\varphi_2(y)$$

$$\begin{tikzcd} & H \times H \arrow[dl, "\pi_1"'] \arrow[d, "m"] \arrow[dr, "\pi_2"] & \\ H & H & H \end{tikzcd}$$

$$(\varphi_1 * \varphi_2) = m_*(\pi_1^*\varphi_1 \cdot \pi_2^*\varphi_2)$$

Where for $f: X \to Y$

$(f^*\psi)(x) = \psi(f(x))$, $(f_*\varphi)(y) = \sum_{x \in f^{-1}(y)} \varphi(x)$

Use $K \subset H$ subgp

to find a comm alg.

$${}^K\mathbb{C}[H]^K = \mathbb{C}[K \backslash H / K] =: \mathcal{H}_{H,K}$$

(Hecke algebra)

$$\begin{tikzcd} & K\backslash H \times_K H/K \arrow[dl, "\pi_1"'] \arrow[d, "m"] \arrow[dr, "\pi_2"] & \\ K\backslash H/K & K\backslash H/K & K\backslash H/K \end{tikzcd}$$

$$\varphi_1 * \varphi_2 = m_*(\pi_1^*\varphi_1 \cdot \pi_2^*\varphi_2)$$ associative alg.

<u>Prop</u> For any rep'n V of H

$\cancel{\mathcal{H}_{H,K}}$ $^KV: \{v \in V \mid k.v = v\} \circlearrowleft \mathcal{H}_{H,K}$

↑ universal way

"Recall"

Frobenius reciprocity

$$\mathrm{Hom}_H(\mathrm{Ind}_K^H W, V) = \mathrm{Hom}_K(W, \mathrm{Res}_H^K V)$$

$$\mathrm{Res}_H^K : \mathrm{Rep}\, H \to \mathrm{Rep}\, K$$

$$\mathrm{Ind}_K^H : \mathrm{Rep}\, K \to \mathrm{Rep}\, H$$

$$W \to \mathbb{C}[H] \otimes_{\mathbb{C}[K]} W$$

"pf" $^KV = \mathrm{Hom}_K(\mathbb{C}_{triv}, \mathrm{Res}_H^K V)$

$= \mathrm{Hom}_H(\mathrm{Ind}_K^H \mathbb{C}_{triv}, V)$

$= \mathrm{Hom}_H(\mathbb{C}[H/K], V) \circlearrowleft \mathcal{H}_{H,K}$

pre composition

$$\mathrm{End}_H(\mathbb{C}[H/K]) \overset{!}{=} \mathcal{H}_{H,K}$$

$$\| \qquad\qquad \|$$

$$^K\mathbb{C}[H/K] = \mathbb{C}[K \backslash H/K]$$

<u>Rmk</u> If K is small, $\mathrm{Rep}\, H$ can survive $^K(-)$, so $\mathcal{H}_{H,K}$ knows a lot about $\mathrm{Rep}\, H$

If K is large, $K \backslash H/K$ is small so $\mathcal{H}_{H,K}$ has better structure (e.g. commutativity)

For $\mathrm{Rep}\, G$ one needs to find the right balance!

Let's translate all this into geometry

$$\begin{tikzcd} & X \times H \arrow[dl, "\pi_1"'] \arrow[d, "a"] \arrow[dr, "\pi_2"] & \\ X & X & H \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & X \times_K H/K \arrow[dl, "\pi_1"'] \arrow[d, "a"] \arrow[dr, "\pi_2"] & \\ X/K & X/K & K\backslash H/K \end{tikzcd}$$

$\varphi \in \mathbb{C}[X]$, $\alpha \in \mathbb{C}[H]$

$\varphi \cdot \alpha = a_*(\pi_1^* \varphi \cdot \pi_2^* \alpha)$ $\qquad \mathbb{C}[X/K] \circlearrowleft \mathcal{H}_{H,K}$

$$\underbrace{\mathrm{Fun}}_{\substack{\| \\ D(\mathrm{Bun}_G)}}(X/K) \circlearrowleft \mathcal{H}_{H,K} = \mathrm{Fun}(K\backslash H/K)$$

$\mathrm{Sat}_G \circlearrowleft D(\mathrm{Bun}_G)$

Goal: Find X, H, K

s.t. $\cdot\ X \circlearrowleft H$

$\cdot\ X/K = \mathrm{Bun}_G = \mathrm{Bun}_G C$

$\rightsquigarrow D(\mathrm{Bun}_G) \circlearrowleft D(K\backslash H/K)$

First work in top'l cat. G connected Lie group

C cmpct. Riemann of genus g.

prop $\mathrm{Bun}_G^{top}(C) \cong L \backslash L^{top}G / L_+^{top}G$

where $L^{top}G = \{D^\times \to G\}$

$L_+^{top}G = \{D \to G\}$

$L_{out}G = \{C^\times \to G\}$

$D^\times = D\setminus\{x\}$

D = disk around x

$D^\times = D \setminus \{x\}$

$C^\times = C \setminus \{x\}$

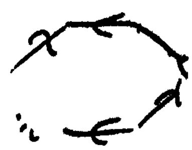
pf) $L^{top}G \cong (P, \alpha, \beta)$ ⇐ claim

where P is a G-bundle on C

$\alpha : P|_D \cong P^0|_D$ ← trivialization

$\beta : P|_{C^\times} \cong P^0|_{C^\times}$ ←

D is contractible ⇝ α exists

$P|_{C^\times}$  $\cong \vee_{2g} S^1$ $[S^1 \to BG]$

$= \pi_1(BG) = \pi_0(G)$

homotopy class is trivial for each S^1
so the whole bundle is trivial

$\gamma = \beta \times \alpha^{-1}_{D^\times} : P^0|_{D^\times} \to P^0|_{D^\times} = D^\times \times G$

$D^\times \to G \in L^{top}G$

$\Rightarrow \gamma \in L^{top}G$, triv on D
triv on $C^\times$

glue them using γ to get P

$L^{top}_+ G$ = trivializations on D

$L^{top}_{out} G$ = trivializations on $C^\times$

Goal: $Bun_G = X/K$

$X = L^{top}_{out}G \backslash L^{top}G$, $H = L^{top}G$

$K = L^{top}_+ G$

In an algebraic category:

Thm 1 (Weil Uniformization)

C smooth proper curve $/ k = \mathbb{F}_q$

$\mathrm{Bun}_G^{(C)}(k) \cong G(k(C)) \backslash G(\mathbb{A}) / G(\mathbb{O})$

where $\mathrm{Bun}_G = \mathrm{Bun}_G(C)$ is moduli of G-bundles on C, $k(C)$ is function field

adele

$\mathbb{A} = \prod^{res}_{x \in C} K_x$, $\quad \mathbb{O} = \prod_{x \in C} \mathcal{O}_x$

with $K_x = k((t_x))$ $\quad \mathcal{O}_x = k[[t_x]]$

Rmk 1 $L^2(G(k(C)) \backslash G(\mathbb{A}) / G(\mathbb{O}))$ is space of automorphic representations! (for unramified case)

$$G(k(C)) \backslash G(K_x) \circlearrowleft G(K_x) \supset G(\mathcal{O}_x)$$

$D(G(\mathcal{O}_x) \backslash G(K_x) / G(\mathcal{O}_x)) \circlearrowleft D(\mathrm{Bun}_G)$ $\quad \forall x \in C$

$\quad \| $

$\mathrm{Sph}_{G,x}$ spherical Hecke category

We finally understand the sense of spectral Decomposition of $D(\mathrm{Bun}_G)$

$\check{G}$?

try to find a neutral Tannakian category $\mathcal{A}_G$

<u>Defn</u> $Gr_G = G(\mathcal{K})/G(\mathcal{O})$ affine Grassmannian

$\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$

$$\boxed{\begin{array}{l} \mathbb{D} = \operatorname{Spec} \mathcal{O} \\ \mathbb{D}^\times = \operatorname{Spec} \mathcal{K} \end{array}}$$

$G(\mathcal{K})$ "=" $\{\text{Maps } \mathbb{D}^\times \to G\}$ loop group

$G(\mathcal{O})$ "=" $\{\text{Maps } \mathbb{D} \to G\}$ affine Kac-Moody group

like the flag variety G/B

Lustig, Drinfeld, Ginzburg, Mirkovic-Virlonen

<u>Thm</u> (Geometric Satake)

$\mathcal{P}_{G(\mathcal{O})}(Gr_G)$ is a neutral Tannakian category

$\mathcal{P}_{G(\mathcal{O})}(Gr_G) \cong \operatorname{Rep} \check{G}$

<u>Rmk</u> $\mathcal{P}$ abelian category of perverse sheaves

D-modules $\overset{RH}{\sim} \mathcal{P}$

works better in char > 0

$D(G(\mathcal{O}_x)\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x))$ has natural $*$ structure

$\cup$

$\mathcal{P}$

Miracles]

① $\mathcal{P}$ is closed under $*$!

② $\phi_1, \phi_2 \in \mathcal{P}$

$$\phi_1 * \phi_2 \cong \phi_2 * \phi_1$$

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2. Basic Geometry of Gr_G

$G = GL_n$

a lattice K^n is an $\mathcal{O}^n$ submodule L

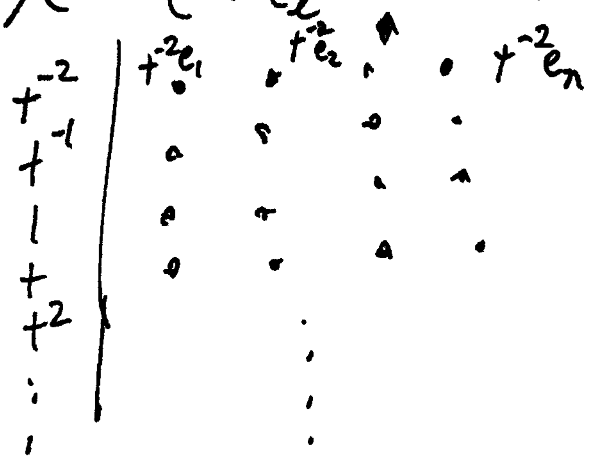
s.t. $t^N \mathcal{O}^n \subset L \subset t^{-N} \mathcal{O}^n$ for some N

prop] $Gr_G \cong \{\text{Lattices in } K^n\}$

$G(K) \curvearrowright \mathcal{O}^n$ lattice transitive

$G(\mathcal{O})$: stabilizer $\rightsquigarrow Gr_G = \{\text{lattice}\}$

K^n $\{t^i e_l \mid i \in \mathbb{Z},\ l = 1, \cdots, n\}$



Imagine $K = \mathbb{Q}$, $\mathcal{O} = \mathbb{Z}$ $\qquad \frac{1}{2^2}\mathbb{Z} \subset \mathbb{Q}$

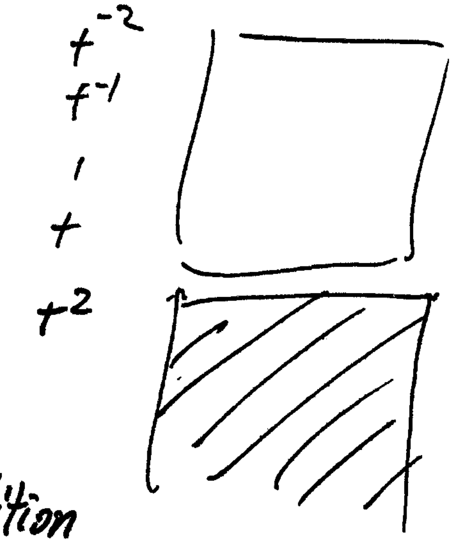
$\mathcal{O}^n$ submodule $\leftrightarrow$ closed under t.

$Gr_G^k \subset \mathbb{C}^{2kn}$

$\|$

$\{$can take $N = k\}$

$k = 2$



closed condition

$Gr_G = \bigcup^k_{k\to\infty} Gr_G^k$ ind-projective variety